A Conformal Basis for Flat Space Amplitudes

SABRINA GONZALEZ PASTERSKI

A scattering basis motivated by asymptotic symmetries?

- **Plane wave** $\Rightarrow$ **Highest weight** scattering

[**arXiv:1701.00049**, **arXiv:1705.01027**, **…** S.Pasterski, S.H. Shao, A. Strominger]

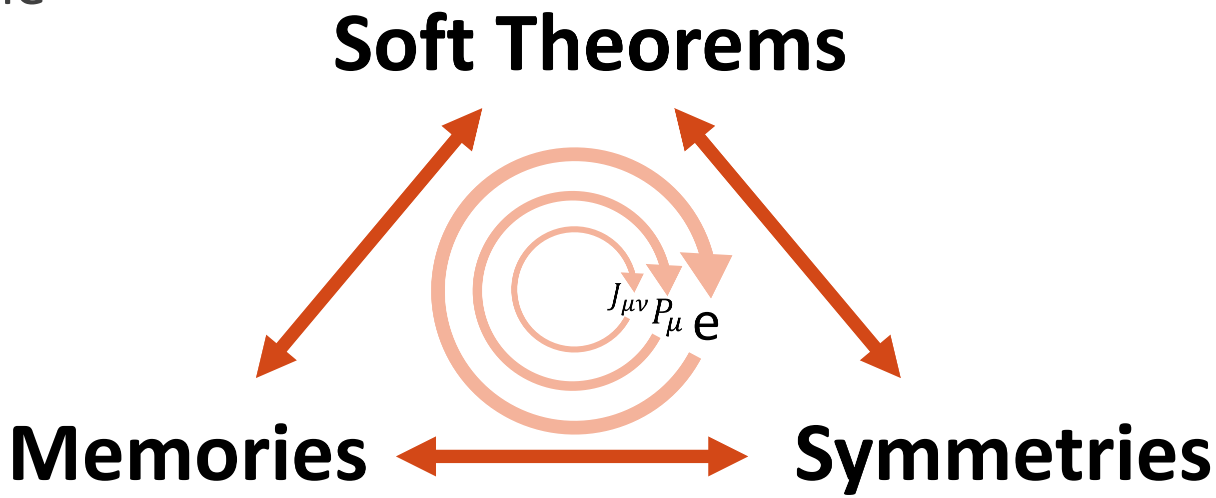
- raison d'être: **Preferred w.r.t. Superrotations**

[**arXiv:1404.4091**, **arXiv:1406.3312**, **arXiv:1502.06120** **…**]

Motivation from Asymptotic Symmetries

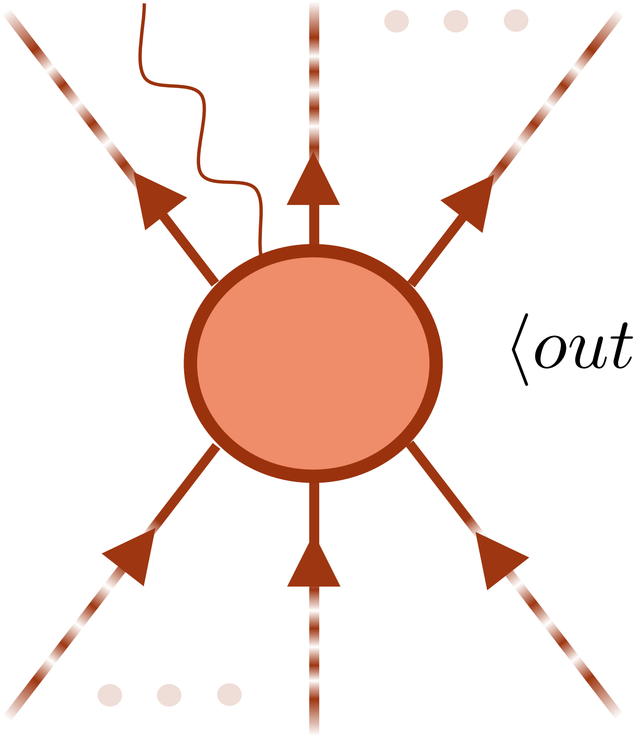
- Recent studies of low energy limits of scattering in gauge theories led to understanding connections between **soft theorems, asymptotic symmetries, & memory effects**

- To the extent that once one vertex was known/hypothesized to be present, the others could be ‘filled in’ thereby reaffirming the conjecture.



Motivation from Asymptotic Symmetries

- Recast **soft theorem** as Ward identity for 'large gauge transformations' that act non-trivially on **boundary** data.



$$\langle out|a_-(q)\mathcal{S}|in\rangle = \left(S^{(0)-} + S^{(1)-}\right)\langle out|\mathcal{S}|in\rangle + \mathcal{O}(\omega)$$

$$S^{(0)-} = \sum_k \frac{(p_k\cdot\epsilon^-)^2}{p_k\cdot q} \qquad S^{(1)-} = -i\sum_k \frac{p_{k\mu}\epsilon^{-\mu\nu}q^\lambda J_{k\lambda\nu}}{p_k\cdot q}$$

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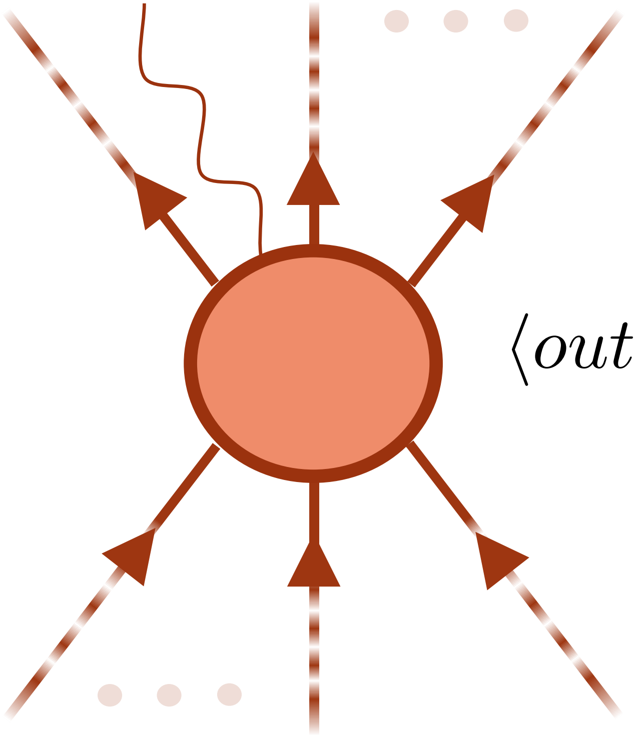

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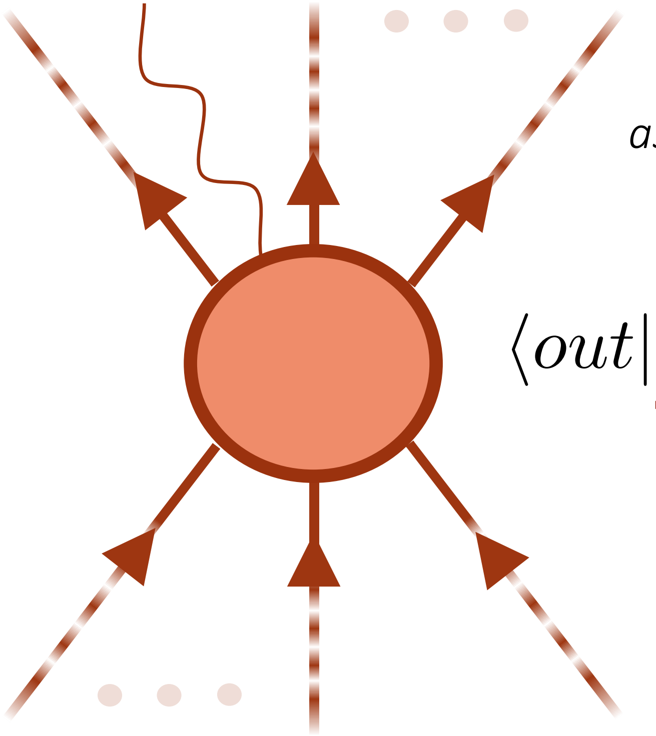

$$e^{iq\cdot x} = e^{-i\omega u - i\omega r(1-\hat{q}\cdot\hat{x})}$$

$\hat{q}_{pos} \Leftrightarrow \hat{q}_{mom}$ & $\lim_{\omega\to 0} \Leftrightarrow \int du$

as Fourier mode of field operator

$$\langle out|a_-(q)\mathcal{S}|in\rangle = \left(S^{(0)-} + S^{(1)-}\right)\langle out|\mathcal{S}|in\rangle + \mathcal{O}(\omega)$$

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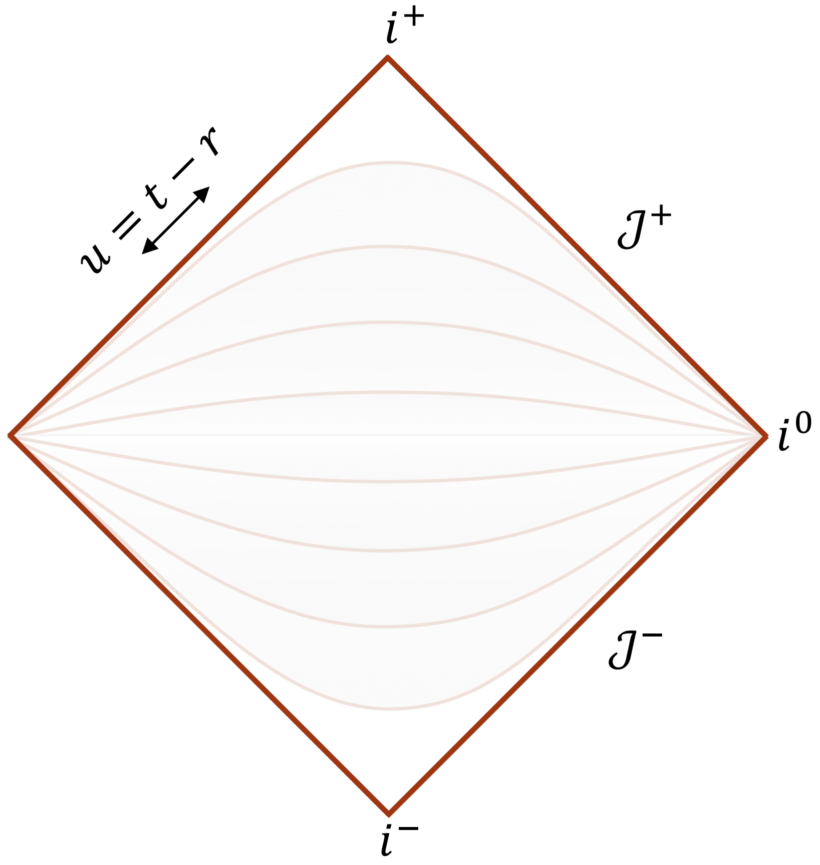
Motivation from Asymptotic Symmetries



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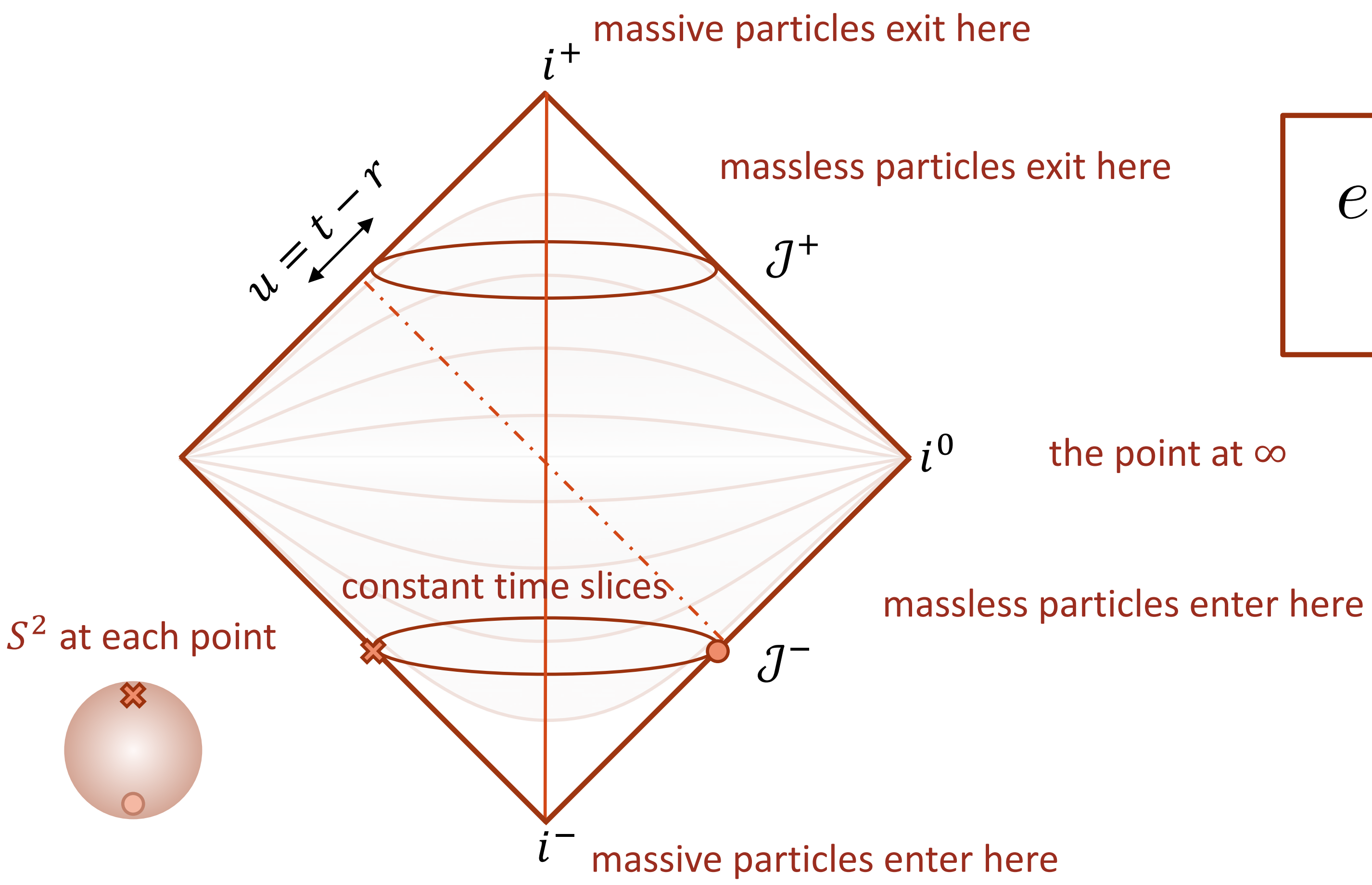
Motivation from Asymptotic Symmetries



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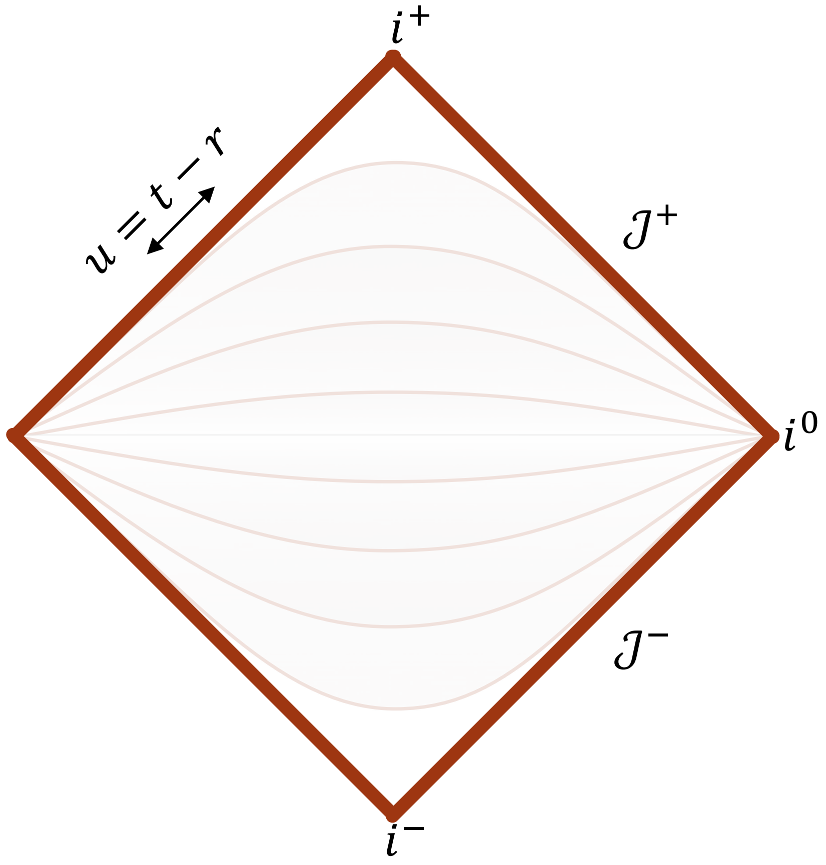
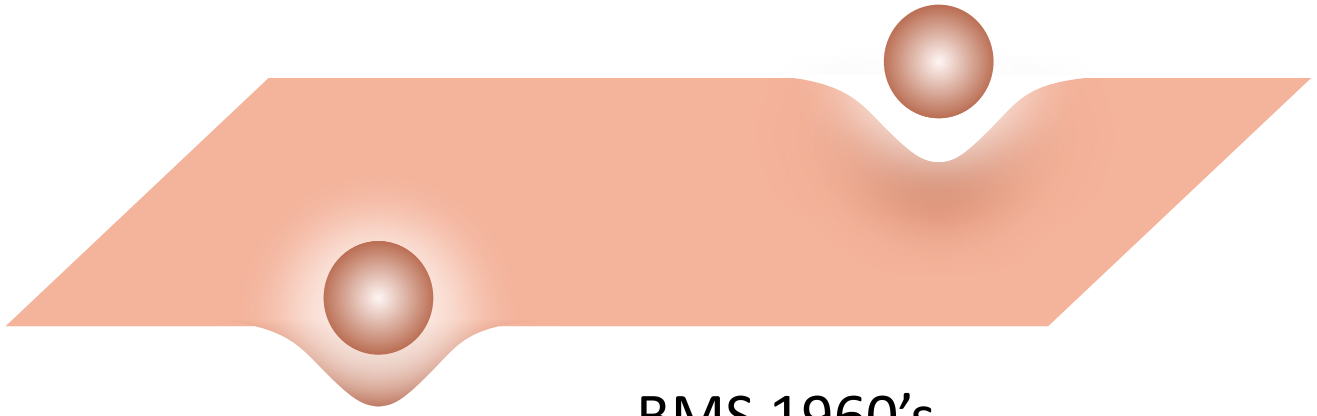
Motivation from Asymptotic Symmetries

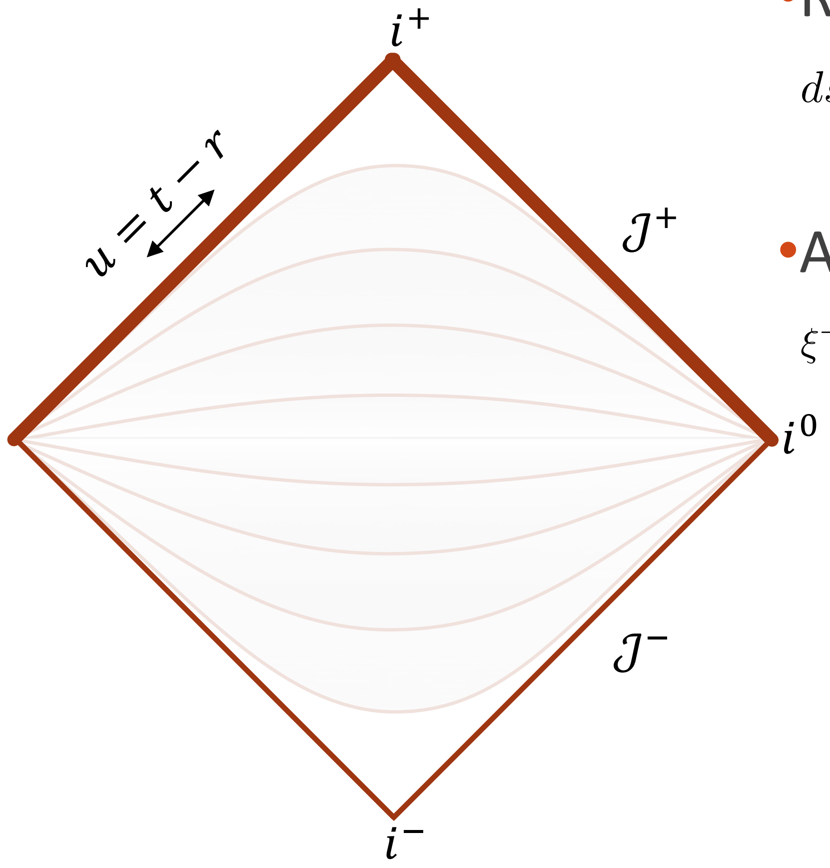


- Interested in set of diffeomorphisms that preserve class of asymptotically flat metrics, characterized by radial fall-off near null infinity



BMS 1960's

Motivation from Asymptotic Symmetries



- Radial Expansion:

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + 2\tfrac{m_B}{r}du^2 \\ &+\left(rC_{zz}dz^2 + D^zC_{zz}dudz + \tfrac{1}{r}\left(\tfrac{4}{3}N_z - \tfrac{1}{4}\partial_z(C_{zz}C^{zz})\right)dudz + c.c.\right) + \ldots \end{aligned}$$

- ASG that preserves this expansion:

$$\begin{aligned} \xi^+ =& (1+\frac{u}{2r})Y^{+z}\partial_z - \frac{u}{2r}D^{\bar{z}}D_zY^{+z}\partial_{\bar{z}} - \frac{1}{2}(u+r)D_zY^{+z}\partial_r + \frac{u}{2}D_zY^{+z}\partial_u + c.c \\ &+ f^+\partial_u - \frac{1}{r}(D^zf^+\partial_z + D^{\bar{z}}f^+\partial_{\bar{z}}) + D^zD_zf^+\partial_r \end{aligned}$$

Coordinate Conventions:

$$z = e^{i\phi}\tan\frac{\theta}{2} \qquad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$$

$$f^+ = f^+(z,\bar{z}) \qquad \partial_{\bar{z}}Y^{+z} = 0$$

Motivation from Asymptotic Symmetries

- Radial Expansion:

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Radiative Data

- ASG that preserves this expansion:

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$$+ f^+\partial_u - \frac{1}{r}(D^zf^+\partial_z + D^{\bar{z}}f^+\partial_{\bar{z}}) + D^zD_zf^+\partial_r$$

Superrotations

Coordinate Conventions:

$$z = e^{i\phi}\tan\frac{\theta}{2} \qquad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$$
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Motivation from Asymptotic Symmetries

- From the **soft theorem** $\Longleftrightarrow$ **Ward identity** perspective the superrotation action corresponds to the subleading soft graviton theorem
- Let us take a closer look at the superrotation vector field near null infinity:

$$\xi^{+}|_{\mathcal{J}^{+}} = Y^{+z}\partial_z + \frac{u}{2} D_z Y^{+z}\partial_u + c.c.$$

  - Notice we have two copies of the Witt algebra since Y is any 2D CKV
  - Also, $u\partial_u$ prefers Rindler energy eigenstates

Motivation from Asymptotic Symmetries

- From the **soft theorem** $\Leftrightarrow$ **Ward identity** perspective the superrotation action corresponds to the subleading soft graviton theorem

- Rather than using the subleading soft factor to establish a 4D Ward identity for this asymptotic symmetry [arXiv:1406.3312] one can massage it to look like a 2D stress tensor insertion [arXiv:1609.00282]

$$T_{zz} \equiv \frac{i}{8\pi G}\int d^2w \frac{1}{z-w} D_w^2 D^{\bar{w}} \int du u \partial_u C_{\bar{w}\bar{w}}$$

$$S^{(1)-} = -i\sum_k \frac{p_{k\mu}\epsilon^{-\mu\nu} q^\lambda J_{k\lambda\nu}}{p_k \cdot q}$$

$$\langle T_{zz}\mathcal{O}_1\cdots\mathcal{O}_n\rangle = \sum_{k=1}^{n}\left[\frac{h_k}{(z-z_k)^2} + \frac{\Gamma^{z_k}_{z_k z_k}}{z-z_k}h_k + \frac{1}{z-z_k}\left(\partial_{z_k} - |s_k|\Omega_{z_k}\right)\right]\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle$$

Weight Conventions:

$$h = \frac{1}{2}(s+1+iE_R) \quad \bar{h} = \frac{1}{2}(-s+1+iE_R)$$

$$\Delta = h + \bar{h} \quad s = h - \bar{h}$$

Motivation: **Recap**

❖ We've highlighted certain aspects of the **soft theorem** $\Longleftrightarrow$ **Ward identity** program relevant to what we will do next:

- How to connect between soft limits and position space ASG's
- The superrotation asymptotic killing vector fields
- The subleading soft theorem as a 2D stress tensor

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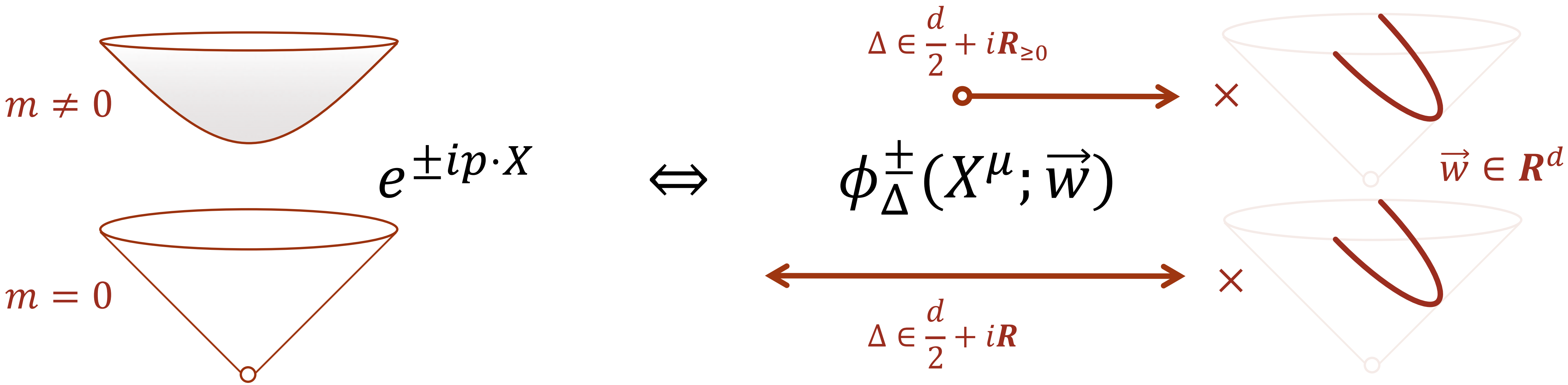
Enhancement from Lorentz to Virasoro

Preferred highest-weight basis

Constructing Highest-Weight Solutions

- Start in arbitrary dimensions $\boldsymbol{R}^{1,d+1}$ [arXiv:1705.01027] then consider examples where d=2 [arXiv:1701.00049 **+ ...**]
- Would like to see if possible (and if so, how) to go back and forth between S-matrix elements in standard **plane wave** versus **highest-weight** bases



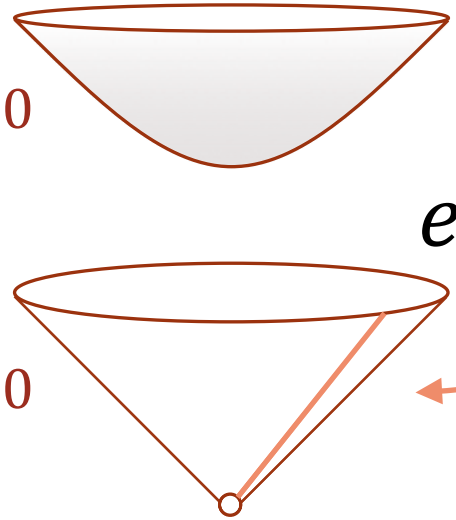
Constructing Highest-Weight Solutions

canonical reference direction when m=0

Principal Continuous Series of SO(1,d+1)

$m \neq 0$

$m = 0$



$\Delta \in \frac{d}{2} + i\boldsymbol{R}_{\geq 0}$

$$e^{\pm i p \cdot X} \Leftrightarrow \phi^{\pm}_{\Delta}(X^{\mu}; \vec{w})$$

$\times$

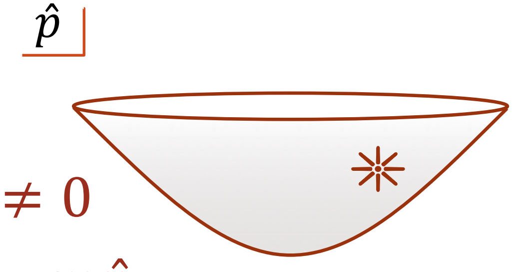
$\times$

$\vec{w} \in \boldsymbol{R}^{d}$

$\Delta \in \frac{d}{2} + i\boldsymbol{R}$

Constructing Highest-Weight Solutions

$$ds^2_{H_{d+1}} = \frac{dy^2 + d\vec{z}\cdot d\vec{z}}{y^2}$$



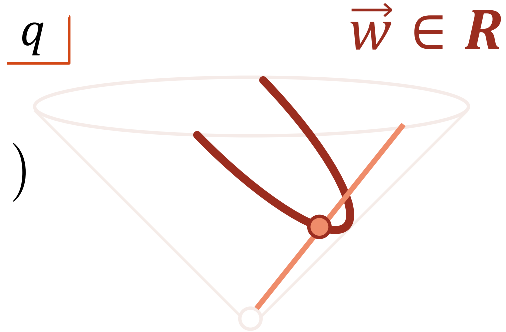
$$\hat{p}(y,\vec{z}) = \left(\frac{1+y^2+|\vec{z}|^2}{2y}\,,\ \frac{\vec{z}}{y}\,,\ \frac{1-y^2-|\vec{z}|^2}{2y}\right)$$

$m \neq 0$

$p = m\hat{p}$

$$\hat{p}^\mu(y',\vec{z}\,') = \Lambda^\mu{}_\nu \hat{p}^\nu$$

$\vec{w} \in \boldsymbol{R}^d$



$$q^\mu(\vec{w}) = \left(1+|\vec{w}|^2\,,\ 2\vec{w}\,,\ 1-|\vec{w}|^2\right)$$

$$q^\mu(\vec{w}\,') = \left|\frac{\partial\vec{w}\,'}{\partial\vec{w}}\right|^{1/d}\ \Lambda^\mu{}_\nu q^\nu(\vec{w})$$

Constructing Highest-Weight Solutions

❖ Using that the Lorentz group SO(1,d+1) in $\boldsymbol{R}^{1,d+1}$ acts as the conformal group on $\boldsymbol{R}^{d}$ define the *massive scalar conformal primary wavefunction* to:

- satisfy the (d+2)-dimensional massive Klein-Gordon equation of mass *m*:

$$\left(\frac{\partial}{\partial X^\nu}\frac{\partial}{\partial X_\nu} - m^2\right)\phi_\Delta(X^\mu;\vec{w}) = 0$$

- transform covariantly as a scalar conformal primary operator in d dimensions under an SO(1,d+1) transformation:

$$\phi_\Delta\left(\Lambda^\mu{}_\nu X^\nu;\vec{w}\,'(\vec{w})\right) = \left|\frac{\partial \vec{w}\,'}{\partial \vec{w}}\right|^{-\Delta/d} \phi_\Delta(X^\mu;\vec{w})$$

Constructing Highest-Weight Solutions

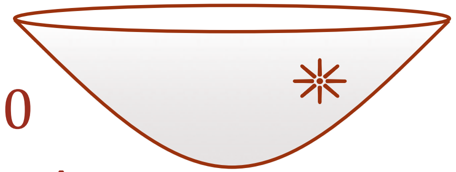
$$G_\Delta(\hat{p}; q) = \frac{1}{(-\hat{p}\cdot q)^\Delta} = \left(\frac{y}{y^2 + |\vec{z} - \vec{w}|^2}\right)^\Delta$$

$$G_\Delta(\hat{p}'; q') = \left|\frac{\partial \vec{w}\,'}{\partial \vec{w}}\right|^{-\Delta/d} G_\Delta(\hat{p}; q)$$

$$ds^2_{H_{d+1}} = \frac{dy^2 + d\vec{z}\cdot d\vec{z}}{y^2}$$

$$\hat{p}(y, \vec{z}) = \left(\frac{1+y^2+|\vec{z}|^2}{2y}, \frac{\vec{z}}{y}, \frac{1-y^2-|\vec{z}|^2}{2y}\right)$$

$\hat{p}$



$m \neq 0$

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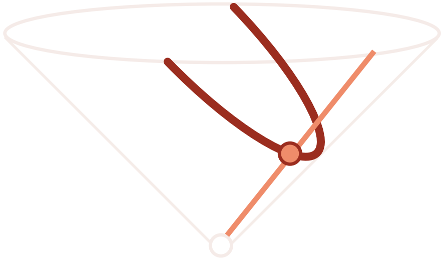
$$\hat{p}^\mu(y', \vec{z}\,') = \underline{\Lambda^\mu{}_\nu \hat{p}^\nu}$$

q

$\vec{w} \in \boldsymbol{R}^d$



$$q^\mu(\vec{w}) = \left(1+|\vec{w}|^2, 2\vec{w}, 1-|\vec{w}|^2\right)$$

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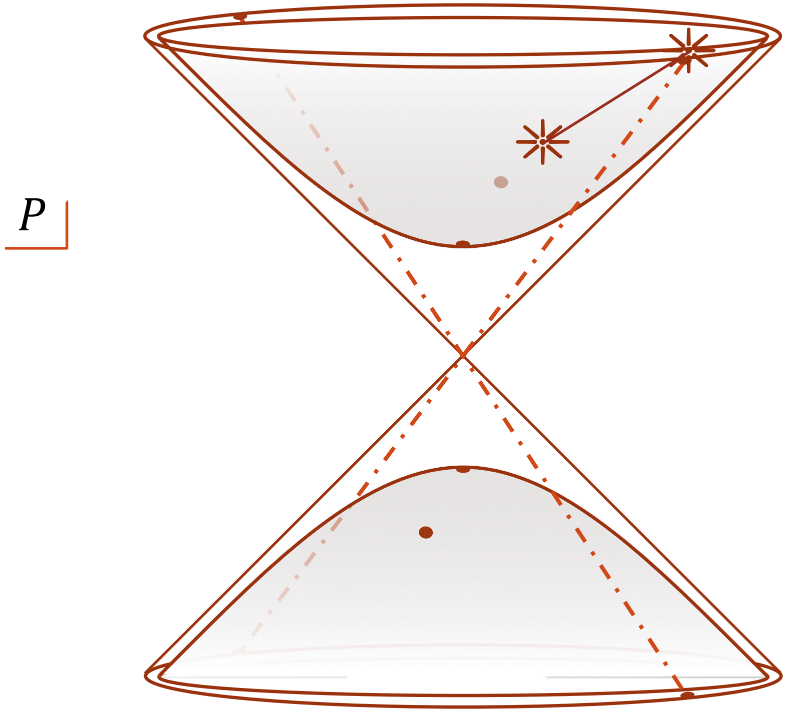
Constructing Highest-Weight Solutions

- The desired properties are met by the convolution:



$$\boxed{\phi^{\pm}_{\Delta}(X^{\mu};\vec{w}) = \int_{H_{d+1}} [d\hat{p}]\, G_{\Delta}(\hat{p};\vec{w})\ \exp\left[\,\pm i m \hat{p}\cdot X\,\right]}$$

- Interpretation as bulk-to-boundary propagation in momentum space
- Have **plane wave** $\Rightarrow$ **highest-weight**, what about reverse?

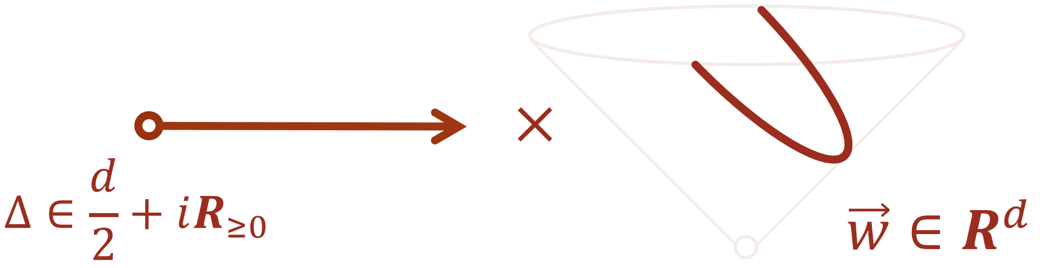
Constructing Highest-Weight Solutions

- If we define the shadow for a scalar as

$$\widetilde{\mathcal{O}}_\Delta(\vec{w}) \equiv \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}\int d^d\vec{w}\,'\frac{1}{|\vec{w}-\vec{w}\,'|^{2(d-\Delta)}}\mathcal{O}_\Delta(\vec{w}\,')$$

- The action on our scalar wavefunctions shows linear dependence between weights Δ and $d-\Delta$

$$\widetilde{\phi^{\pm}_{\Delta}}(X;\vec{w}) = \phi^{\pm}_{d-\Delta}(X;\vec{w})$$



Constructing Highest-Weight Solutions

- The orthogonality conditions

$$\int_{-\infty}^{\infty} d\nu\, \mu(\nu) \int d^d\vec{w}\, G_{\frac{d}{2}+i\nu}(\hat{p}_1;\vec{w}) G_{\frac{d}{2}-i\nu}(\hat{p}_2;\vec{w}) = \delta^{(d+1)}(\hat{p}_1,\hat{p}_2)$$

$$\int_{H_{d+1}} [d\hat{p}]\, G_{\frac{d}{2}+i\nu}(\hat{p};\vec{w}_1) G_{\frac{d}{2}+i\bar{\nu}}(\hat{p};\vec{w}_2) =$$

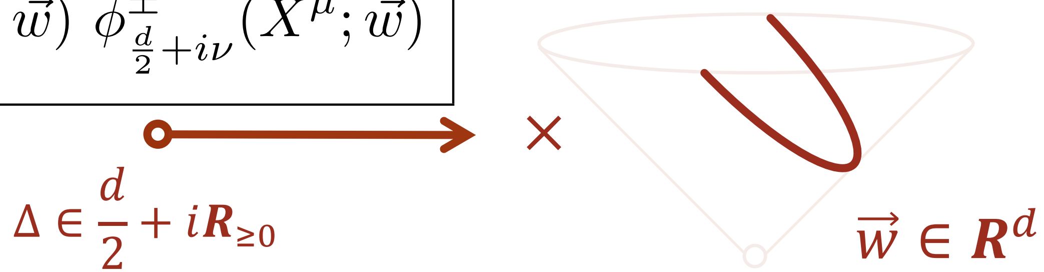
$$2\pi^{d+1} \frac{\Gamma(i\nu)\Gamma(-i\nu)}{\Gamma(\frac{d}{2}+i\nu)\Gamma(\frac{d}{2}-i\nu)} \delta(\nu+\bar{\nu})\delta^{(d)}(\vec{w}_1-\vec{w}_2) + 2\pi^{\frac{d}{2}+1} \frac{\Gamma(i\nu)}{\Gamma(\frac{d}{2}+i\nu)} \delta(\nu-\bar{\nu}) \frac{1}{|\vec{w}_1-\vec{w}_2|^{2(\frac{d}{2}+i\nu)}}$$

$$\mu(\nu) = \frac{\Gamma(\frac{d}{2}+i\nu)\Gamma(\frac{d}{2}-i\nu)}{4\pi^{d+1}\Gamma(i\nu)\Gamma(-i\nu)}$$

[arXiv:1404.5625]

- Imply we can go in the opposite direction **highest-weight** $\Rightarrow$ **plane wave**

$$\boxed{e^{\pm i m \hat{p}\cdot X} = 2\int_0^{\infty} d\nu\, \mu(\nu) \int d^d\vec{w}\; G_{\frac{d}{2}-i\nu}(\hat{p};\vec{w})\; \phi^{\pm}_{\frac{d}{2}+i\nu}(X^\mu;\vec{w})}$$



Constructing Highest-Weight Solutions

- And we see that the Klein-Gordon inner product

$$(\Phi_1, \Phi_2) = -i \int d^{d+1}X^i \; [\Phi_1(X)\,\partial_{X^0}\Phi_2^*(X) - \partial_{X^0}\Phi_1(X)\,\Phi_2^*(X)]$$

evaluated between solutions in our basis is of a distributional nature

$$\left(\phi^{\pm}_{\frac{d}{2}+i\nu_1}(X^\mu;\vec{w}_1), \phi^{\pm}_{\frac{d}{2}+i\nu_2}(X^\mu;\vec{w}_2)\right)$$

$$= \pm \frac{2^{d+3}\pi^{2d+2}}{m^d} \frac{\Gamma(i\nu_1)\Gamma(-i\nu_1)}{\Gamma(\frac{d}{2}+i\nu_1)\Gamma(\frac{d}{2}-i\nu_1)} \delta(\nu_1-\nu_2)\,\delta^{(d)}(\vec{w}_1-\vec{w}_2)$$

$$\pm \frac{2^{d+3}\pi^{\frac{3d}{2}+2}}{m^d} \frac{\Gamma(i\nu_1)}{\Gamma(\frac{d}{2}+i\nu_1)} \delta(\nu_1+\nu_2) \frac{1}{|\vec{w}_1-\vec{w}_2|^{2(\frac{d}{2}+i\nu_1)}} \longrightarrow \times$$

$\Delta \in \frac{d}{2} + i\boldsymbol{R}_{\geq 0}$

$\vec{w} \in \boldsymbol{R}^d$

Massless Highest-Weight Solutions

- We have been using properties of the **bulk-to-boundary propagator** on the momentum space hyperboloid (**conformal covariance, orthogonality, completeness**) to convert between plane waves and highest weight solutions.
- By forming the combination $\omega = \dfrac{m}{2y}$ we can further use the boundary behavior of G_Δ to explore the massless analog:

$$G_\Delta(y,\vec{z};\vec{w}) \underset{m\to 0}{\longrightarrow} \pi^{\frac{d}{2}} \frac{\Gamma(\Delta-\frac{d}{2})}{\Gamma(\Delta)} y^{d-\Delta}\delta^{(d)}(\vec{z}-\vec{w}) + \frac{y^\Delta}{|\vec{z}-\vec{w}|^{2\Delta}} + \cdots$$

$$\phi^{\pm}_{\frac{d}{2}+i\nu}(X;\vec{w}) \underset{m\to 0}{\longrightarrow} \left(\frac{m}{2}\right)^{-\frac{d}{2}-i\nu} \frac{\pi^{\frac{d}{2}}\Gamma(i\nu)}{\Gamma(\frac{d}{2}+i\nu)} \int_0^\infty d\omega\, \omega^{\frac{d}{2}+i\nu-1} e^{\pm i\omega q(\vec{w})\cdot X} + \left(\frac{m}{2}\right)^{-\frac{d}{2}+i\nu} \int d^d\vec{z} \frac{1}{|\vec{z}-\vec{w}|^{2(\frac{d}{2}+i\nu)}} \int_0^\infty d\omega \omega^{\frac{d}{2}-i\nu-1}\, e^{\pm i\omega q(\vec{z})\cdot X}$$

Massless Highest-Weight Solutions

- The first term satisfies the desired properties of a massless highest weight solution on its own.
- It can be identified as a Mellin transform of the energy, in which the reference direction is the same as the momentum.

$$\boxed{\varphi^{\pm}_{\Delta}(X^\mu;\vec{w}) \equiv \int_0^\infty d\omega\,\omega^{\Delta-1}\,e^{\pm i\omega q\cdot X-\epsilon\omega} = \frac{(\mp i)^{\Delta}\Gamma(\Delta)}{(-q(\vec{w})\cdot X \mp i\epsilon)^{\Delta}}}$$

$$e^{\pm i\omega q\cdot X-\epsilon\omega} = \int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\,\omega^{-\frac{d}{2}-i\nu}\frac{(\mp i)^{\frac{d}{2}+i\nu}\Gamma(\frac{d}{2}+i\nu)}{(-q\cdot X\mp i\epsilon)^{\frac{d}{2}+i\nu}}\,,$$

$$\omega > 0 \xleftrightarrow[\Delta \in \frac{d}{2}+i\boldsymbol{R}]{} \times$$

$\vec{w} \in \boldsymbol{R}^d$

Massless Highest-Weight Solutions

- Photon

$$\left(\frac{\partial}{\partial X^\sigma}\frac{\partial}{\partial X_\sigma}\delta^\mu_\nu - \frac{\partial}{\partial X^\nu}\frac{\partial}{\partial X_\mu}\right) A^{\Delta\pm}_{\mu a}(X^\rho;\vec{w}) = 0$$

$$A^{\Delta\pm}_{\mu a}\left(\Lambda^\rho{}_\nu X^\nu;\vec{w}\,'(\vec{w})\right) = \frac{\partial w^b}{\partial w'^a}\left|\frac{\partial \vec{w}'}{\partial \vec{w}}\right|^{-(\Delta-1)/d} \Lambda_\mu{}^\sigma A^{\Delta\pm}_{\sigma b}(X^\rho;\vec{w})$$

- Graviton

$$\partial_\sigma\partial_\nu h^\sigma{}_{\mu;a_1a_2} + \partial_\sigma\partial_\mu h^\sigma{}_{\nu;a_1a_2} - \partial_\mu\partial_\nu h^\sigma{}_{\sigma;a_1a_2} - \partial^\rho\partial_\rho h_{\mu\nu;a_1a_2} = 0$$

$$h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2} = h^{\Delta,\pm}_{\mu_2\mu_1;a_1a_2}\,,\qquad h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2} = h^{\Delta,\pm}_{\mu_1\mu_2;a_2a_1}\,,\qquad \delta^{a_1a_2}h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2} = 0$$

$$h^{\Delta,\pm}_{\mu_1\mu_2;a_1a_2}\left(\Lambda^\rho{}_\nu X^\nu;\vec{w}\,'(\vec{w})\right) = \frac{\partial w^{b_1}}{\partial w'^{a_1}}\frac{\partial w^{b_2}}{\partial w'^{a_2}}\left|\frac{\partial \vec{w}'}{\partial \vec{w}}\right|^{-(\Delta-2)/d} \Lambda_{\mu_1}{}^{\sigma_1}\Lambda_{\mu_2}{}^{\sigma_2} h^{\Delta,\pm}_{\sigma_1\sigma_2;b_1b_2}(X^\rho;\vec{w})$$

$$\Delta \in \frac{d}{2} + i\boldsymbol{R} \qquad \times$$

$\vec{w} \in \boldsymbol{R}^d$

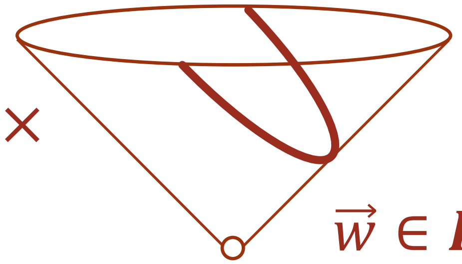
Massless Highest-Weight Solutions

- The shadow is linearly independent.
- Demanding conformal profile fixes residual gauge transformations but within gauge equivalence class can return to Mellin representative.

$$A^{\Delta,\pm}_{\mu a}(X^\mu;\vec{w}) = \frac{\partial_a q_\mu}{(-q\cdot X \mp i\epsilon)^\Delta} + \frac{\partial_a q\cdot X}{(-q\cdot X \mp i\epsilon)^{\Delta+1}} q_\mu$$

$$-\,\textit{const.}\ \frac{\partial}{\partial X^\mu}\left(\frac{\partial_a q\cdot X}{(-q\cdot X \mp i\epsilon)^\Delta}\right)$$

$\Delta \in \frac{d}{2} + i\boldsymbol{R}$ $\times$



$\vec{w} \in \boldsymbol{R}^d$

Amplitude Transforms

- It is useful to point out that the above transforms can be applied directly to the S-matrix elements.

Massive scalar

$$\widetilde{\mathcal{A}}(\Delta_i, \vec{w}_i'(\vec{w}_i)) = \prod_{k=1}^{n} \left| \frac{\partial \vec{w}_k'}{\partial \vec{w}_k} \right|^{-\Delta_k/d} \widetilde{\mathcal{A}}(\Delta_i, \vec{w}_i)$$

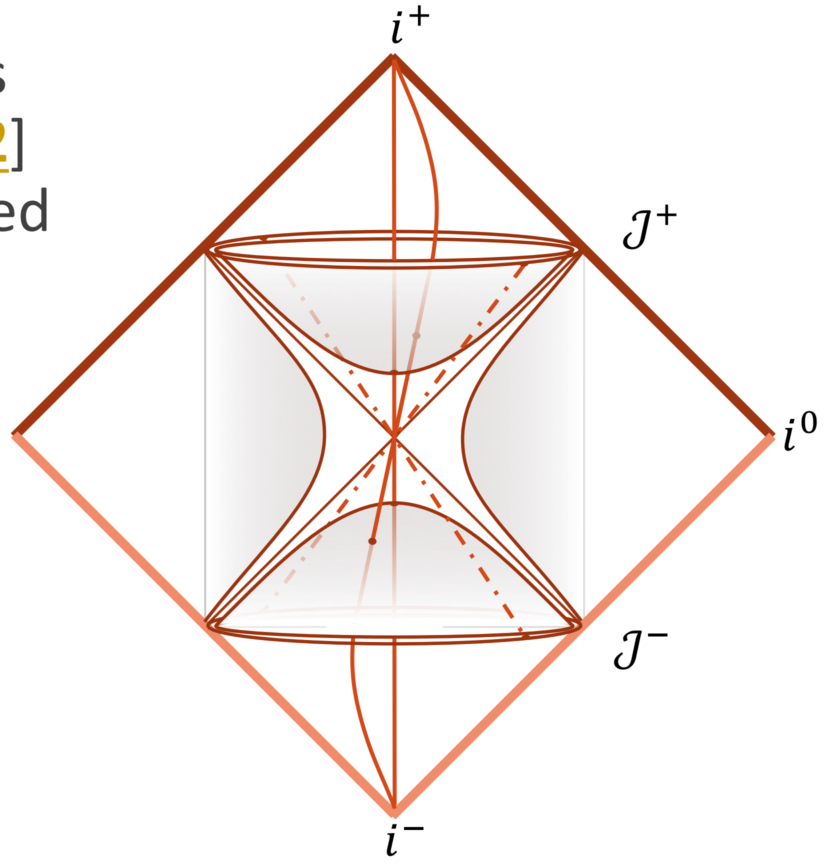
$$\widetilde{\mathcal{A}}(\Delta_i, \vec{w}_i) \equiv \prod_{k=1}^{n} \int_{H_{d+1}} [d\hat{p}_k]\, G_{\Delta_k}(\hat{p}_k; \vec{w}_k)\;\; \mathcal{A}(\pm m_i \hat{p}_i^{\mu})$$

$m = 0$

$$\widetilde{\mathcal{A}}(\Delta_i, \vec{w}_i) \equiv \prod_{k=1}^{n} \int_0^{\infty} d\omega_k \omega_k^{\Delta-1}\, \mathcal{A}(\pm \omega_k q_k^{\mu})$$

Amplitude Transforms

- Note that transforming momentum space amplitudes directly, is an alternative to previous approaches [hep-th/0303006,arXiv:1609.00732] towards flat space holography, which have looked at a foliation of Minkowski space to reproduce AdS/CFT, dS/CFT on each slice.



Amplitude Transforms

- From [arXiv:1705.01027], summarized here, we know that we can equivalently consider plane wave or highest-weight scattering states on the principal continuous series.

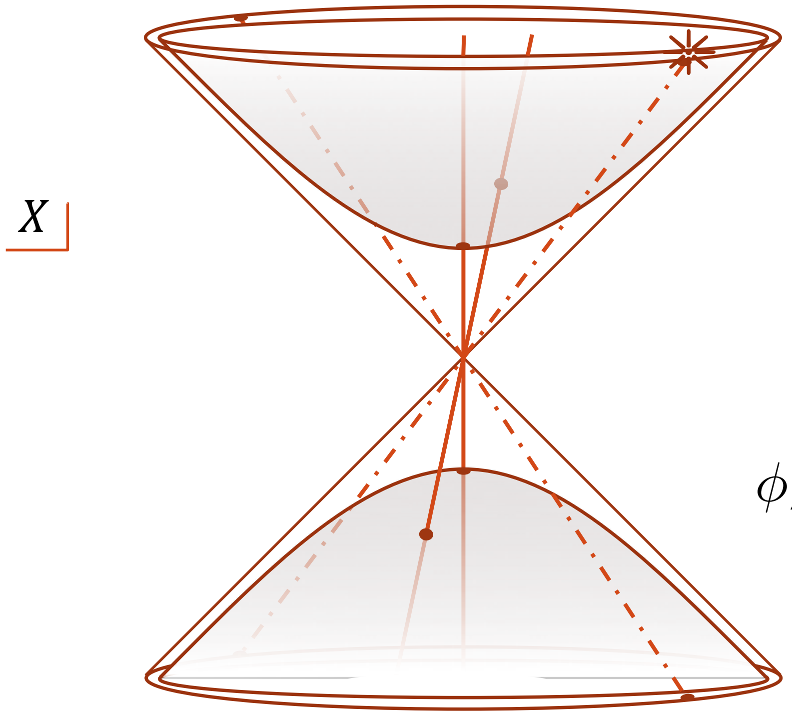
  - *So the basis motivated by the subleading soft-theorem is **okay** but is it **useful?***

- Look at d=2 examples:

  - Massive scalar 3pt near-extremal decay [arXiv:1701.00049]
  - MHV Mellin amplitudes

Massive Scalar 3pt

- For d=2, we use the projective coordinate w, for the celestial sphere CS^2 at the boundary of the lightcone from the origin. w undergoes mobius transformations when the spacetime undergoes Lorentz transformations



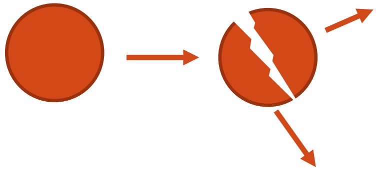
$$w = \frac{X^1 + iX^2}{X^0 + X^3} \qquad w \to \frac{aw+b}{cw+d}$$

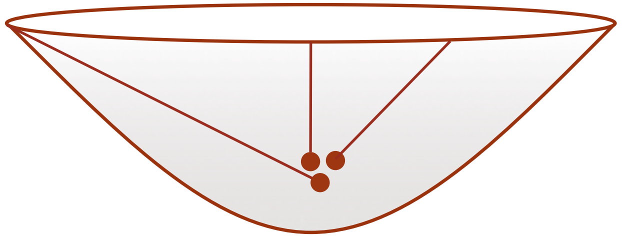
- The highest weight states now look like quasi-primaries under SL(2,C)

$$\phi_{\Delta,m}\left(\Lambda^{\mu}{}_{\nu}X^{\nu}; \frac{aw+b}{cw+d}, \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}\right) = |cw+d|^{2\Delta}\phi_{\Delta,m}\left(X^{\mu}; w, \bar{w}\right)$$

Massive Scalar 3pt

- Lorentz covariance is built into the definition of the basis. If non-zero/finite 4D Lorentz covariance dictates 2D-correlator form.
- The behavior of low-point "correlation functions" is strongly dictated by momentum conservation in the bulk. Special scattering configurations can be used to get Witten diagram-like results.





$$2(1+\epsilon)m \to m + m$$

$$\tilde{\mathcal{A}}(w_i,\bar{w}_i) = \frac{i2^{\frac{9}{2}}\pi^6\lambda\Gamma(\frac{\Delta_1+\Delta_2+\Delta_3-2}{2})\Gamma(\frac{\Delta_1+\Delta_2-\Delta_3}{2})\Gamma(\frac{\Delta_1-\Delta_2+\Delta_3}{2})\Gamma(\frac{-\Delta_1+\Delta_2+\Delta_3}{2})\sqrt{\epsilon}}{m^4\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)|w_1-w_2|^{\Delta_1+\Delta_2-\Delta_3}|w_2-w_3|^{\Delta_2+\Delta_3-\Delta_1}|w_3-w_1|^{\Delta_3+\Delta_1-\Delta_2}} + \mathcal{O}(\epsilon)$$

[arXiv:1701.00049]

MHV Mellin

- Momentum conservation strongly dictates the form of low point Mellin amplitudes. If we think of correlation functions of Mellin operators, we see the contact nature of the two point function already from the scalar Mellin modes:

$$\mathfrak{a}_\lambda(\hat{q}) \equiv \int_0^\infty d\omega\ \omega^{i\lambda} a(\omega, \hat{q}) \qquad \langle 0|\mathfrak{a}_{\lambda'}(\hat{q}')\mathfrak{a}^\dagger_\lambda(\hat{q})|0\rangle = (2\pi)^4 \delta(\lambda - \lambda')\delta^{(2)}(w_1 - w_2)$$

- For MHV amplitudes (and any theory with scale invariance) one finds that the Mellin transformed amplitudes have a conservation-of-weight

$$\tilde{A}_{\Delta_1,\cdots,\Delta_n}(w_i, \bar{w}_i) \equiv \prod_{k=1}^n \int_0^\infty d\omega_k \omega_k^{i\lambda_k}\ \mathcal{A}(\omega_k q_k^\mu) \longrightarrow \tilde{A}_n = \prod_{k=1}^n \int_0^\infty d\omega_k \omega_k^{i\lambda_k} \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}\delta^4(\sum_k p_k)$$

$$\langle ij\rangle = 2\sqrt{\omega_i\omega_j}(w_i - w_j) \qquad \tilde{A} \propto \int_0^\infty ds s^{i\sum \lambda_k - 1} = 2\pi\delta(\sum \lambda_k)$$

MHV Mellin

- Once you tell me the directions of scattering, the frequencies in the mellin integral get fixed, ie the momentum conserving delta functions localize the frequency integrals (and then some). For a $2 \to 2$ process with helicities $(- - + +)$

$$\tilde{\mathcal{A}}_4 = (-1)^{1+i\lambda_2+i\lambda_3}\,\frac{\pi}{2}\left[\frac{\eta^5}{1-\eta}\right]^{1/3} \delta(\mathrm{Im}[\eta])$$

$$\times\, \delta(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\prod_{i<j}^{4} z_{ij}^{h/3-h_i-h_j}\,\bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j}$$

$$h^- = \frac{i}{2}\lambda \qquad \bar{h}^- = 1+\frac{i}{2}\lambda$$

$$h^+ = 1+\frac{i}{2}\lambda \qquad \bar{h}^+ = \frac{i}{2}\lambda$$

MHV Mellin

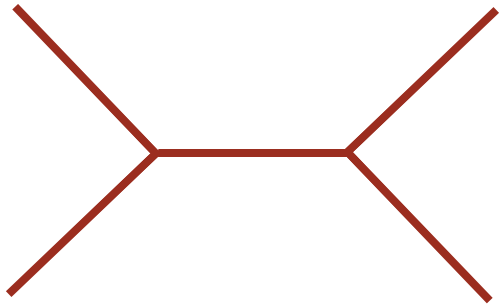
- On-shell + momentum conserving kinematics restrict 2 → 2 reference directions to lie on a circle within the celestial sphere
- MHV 3pt has no support in (1,3) signature but can analytically continue to (2,2) signature with independent real coordinates

$$\tilde{\mathcal{A}}_3(\lambda_i; z_i, \bar{z}_i) = \pi(-1)^{i\lambda_1}\mathrm{sgn}(z_{23})\mathrm{sgn}(z_{13})\,\delta(\sum_i \lambda_i)\frac{\delta(\bar{z}_{13})\delta(\bar{z}_{12})}{z_{12}^{-1-i\lambda_3}\,z_{23}^{1-i\lambda_1}\,z_{13}^{1-i\lambda_2}}$$

MHV Mellin

- One can then use a slightly modified BCFW, combined with Mellin and inverse Mellin transforms to check consistency of the 4 pt result.

$$\tilde{\mathcal{A}}_{--++}(\lambda_i, z_i, \bar{z}_i) = |1-z| \left(\frac{\bar{z}_{24}}{\bar{z}_{14}}\right)^{2+i\lambda_1} \left(\frac{z_{13}}{z_{14}}\right)^{2+i\lambda_4}$$
$$\times \int_{-\infty}^{\infty} \frac{dU}{U} \int_{-\infty}^{\infty} \frac{d\lambda_P}{2\pi} \int dz_P d\bar{z}_P \tilde{\mathcal{A}}_{--+}(\lambda_1, \lambda_2, \lambda_P; \tilde{z}_j, \tilde{\bar{z}}_j) \tilde{\mathcal{A}}_{++-}(\lambda_3, \lambda_4, -\lambda_P; \tilde{z}_j, \tilde{\bar{z}}_j)$$



MHV Mellin

- Still much more singular than one might hope to have if the superrotation-inspired putative CFT_2 dual could actually be manifested… **Options?**

  - *Shadow*

$$\mathcal{O}^{+}_{i\lambda}(w,\bar{w}) = \phi^{+}_{i\lambda}(w,\bar{w}) + C_{+,\lambda}\int d^2z \frac{1}{(z-w)^{2+i\lambda}(\bar{z}-\bar{w})^{i\lambda}}\phi^{-}_{-i\lambda}(z,\bar{z})$$

$$\mathcal{O}^{-}_{i\lambda}(w,\bar{w}) = \phi^{-}_{i\lambda}(w,\bar{w}) + C_{-,\lambda}\int d^2z \frac{1}{(z-w)^{i\lambda}(\bar{z}-\bar{w})^{2+i\lambda}}\phi^{+}_{-i\lambda}(z,\bar{z})$$

- Have Mellin & Mellin + Shadow as equally good bases for scattering
  - Give ‘standard’ non-contact 2pt terms, 4pt also promising

MHV Mellin

- More curiously, issue of what linear combination of bases to use connects back to soft theorems initiating this investigation

$$\mathbf{a}_- \equiv a_-(\omega\hat{x}) - \frac{1}{2\pi}\int d^2w \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}} a_+(\omega\hat{y})$$

- The mode combination that decouples in the soft limit (ie zero soft factor) is precisely a linear combination of Mellin and Mellin+shadow in the limit where $\mathrm{Im}\,\Delta = 0$.

- Also single helicity basis becomes more natural.

A scattering basis motivated by asymptotic symmetries?

- Asymptotic symmetry / soft physics investigation motivated by desire to constrain S-matrix via promoting more symmetries as ‘physical’
- Led to a superrotation iteration that hinted at Lorentz → Virasoro + putative stress tensor via subleading soft factor
- Find that the states preferred by this action indeed form a basis for single particle scatterers.
- Secret hope for OPE ⟷ Amplitude recursion relation statement?
- Intermediate obstacles to fleshing out the putative dual seem to at least offer resolutions to some issues that arose in the study of the soft sector alone.

A Conformal Basis for Flat Space Amplitudes

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